



A PARAMETRIC METHOD OF CONSTRUCTING POINCARÉ MAPPINGS IN HYDRODYNAMIC SYSTEMS†

A. G. PETROV

Moscow

e-mail: petrov@ipmnet.ru

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The plane-parallel motion of the particles of an incompressible medium reduces to an investigation of a Hamilton system. The stream function is a Hamilton function. A Hamilton function, which depends periodically on time and corresponds to the agitation of an incompressible medium in a domain which varies periodically with time, is considered. This agitation of the medium is due to dynamic chaos. The transition to dynamic chaos is described by investigating the location of the Lagrangian particles over time intervals which are multiples of the period (Poincaré points (PP)). The set of PP can be obtained using a Poincaré mapping in the phase flow. The method which has been developed is used to investigate the plane-parallel motion of the particles in an incompressible fluid in a thin layer, bounded by a flat bottom, rectilinear side walls and an upper boundary which is deformed according to a specified periodic law. The motion of the particles is determined from Hamilton's system of equations. The Hamiltonian (the stream function) is found in the thin-layer approximation and depends on two dimensionless parameters: the amplitude of deformation and the tangential velocity in the deforming boundary. The characteristic boundary, which separates the domain of the chaotic motion of the PP from the domain of ordered motion, is determined numerically in the domain of the two parameters. The topological structure of the phase trajectories up to the transition to chaotic conditions is analysed using the Poincaré mapping, found with an accuracy up to the third order with respect to the amplitude. The phase trajectories of the PP, found analytically, turn out to be close to the trajectories of the initial Hamilton system, determined numerically. The mapping found in the domain of the two dimensionless parameters enables one to determine, qualitatively, the boundary of the transition to chaos. © 2003 Elsevier Science Ltd. All rights reserved.

It is well known [1, 2] that generating functions enable one to construct the mappings which are achieved by the phase flow of a Hamilton system. An alternative method of constructing these mappings in parametric form is proposed below. A mapping $(X_0, Y_0) \rightarrow (X, Y)$, with a Jacobian equal to unity, is written out parametrically in the form $(X_0, Y_0) \rightarrow (x, y) \rightarrow (X, Y)$ and expressed in terms of the function $\Psi(x, y)$ of the parameters x and y . An equation of the Jacobi type is obtained for a mapping which is achieved by the phase flow of a Hamilton system. In the case of Hamilton systems of standard form, the solution for Ψ is represented in the form of a power series which converges with respect to a small parameter.

The parametric method has a number of advantages over the classical method of generating functions in exactly integrable cases. There is significantly better convergence of the power series for the function Ψ , the corresponding approximations are found to be simpler, and the accuracy is significantly higher compared with the analogous classical results. It is proved that, in the case of an autonomous Hamilton system, the power series in a small parameter for Ψ does not contain even powers.

1. INTRODUCTION

The Cauchy problem

$$\begin{aligned} X &= H_Y, & Y &= -H_X, & X(t_0) &= X_0, & Y(t_0) &= Y_0 \\ H_X &= \partial H / \partial X, & H_Y &= \partial H / \partial Y \end{aligned} \quad (1.1)$$

for a Hamilton system of equations with n degrees of freedom is considered, where $H(t, X, Y) = H(t + T, X, Y)$ is an arbitrary, sufficiently smooth, T -periodic function and X, Y are n -dimensional vectors.

At present there is no generally accepted definition of chaotic motions. In the case of system with one degree of freedom, it is possible to give a simple, geometrical definition of chaos although it is not

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completely rigorous mathematically. In this case, the Hamiltonian of the system can be treated as the stream function of the flow of an incompressible medium. The Hamilton system (1.1) determines the motion of the Lagrangian particles of the medium. We will now introduce several concepts which are required in order to determine the chaos of Lagrangian particles in system with one degree of freedom.

In the trajectory $\mathbf{R}\{\mathbf{X}(t, t_0, \mathbf{X}_0, \mathbf{Y}_0), \mathbf{Y}(t, t_0, \mathbf{X}_0, \mathbf{Y}_0)\}$, which is determined from the solution of system (1.1), we consider the location of points after time intervals which are multiples of the period $\mathbf{R}_m\{\mathbf{X}(t_n, t_0, \mathbf{X}_0, \mathbf{Y}_0), \mathbf{Y}(t_n, t_0, \mathbf{X}_0, \mathbf{Y}_0)\}$, $t_n = t_0 + Tn$, $n = 0, \pm 1, \pm 2, \dots$. The points are called *Poincaré points* (PP). The set of PP is the track of a particle in the case of filming with a frequency of the motion picture frames corresponding to the period T . The PP form a denumerable set of points in a plane which, generally speaking, depends on t_0, X_0, Y_0 . If the set of PP belongs to a one-dimensional line, this line is called an *invariant curve*. In the case of ordered motion, all of the sets of PP form a family of invariant curves. In the hydrodynamic example considered below (in Section 9), this case is illustrated for the first four phase portraits of the PP. In the second case, the set of PP occupies a two-dimensional domain. We shall call this case the chaotic motion of PP. Examples of calculations of chaotic motions are presented in the last six phase portraits of PP.

In the case when the Hamiltonian depends periodically on time, the set of PP can be calculated using the recurrence formulae $\mathbf{R}_n = P_{t_0}^T(\mathbf{R}_{n-1})$, where $P_{t_0}^{\Delta t}(\Delta t = t - t_0)$ is a mapping of system (1.1) in the phase flow or simply the solution of problem (1.1). A mapping after a period, $P_{t_0}^T$, is called a Poincaré mapping.

Note that the definition of chaotization which has been introduced is independent of t_0 . Actually, when t_0 is changed to $t_0 + \Delta t$, $0 < \Delta t < T$, the set of PP is transformed using the continuous mapping $P_{t_0}^{\Delta t}$. At the same time, the one-dimensional line or the two-dimensional domain transfer respectively into a one-dimensional line or a two-dimensional domain. If $\Delta t = kT$, that is, it is a multiple of the period, the mapping $P_{t_0}^{\Delta t} = P_{t_0}^{kT}$ transforms the set of PP into itself. In the case of an arbitrary value, Δt can be represented in the form $\Delta t = kT + \Delta t'$, $0 < \Delta t' < T$. The mapping $P_{t_0}^{\Delta t}$ is identical to the mapping $P_{t_0}^{\Delta t'}$ and does not change the topological structure of the PP.

Henceforth, to be specific, we will choose $t_0 = 0$ and we shall omit the subscript and superscript in the notation for a Poincaré mapping: $P = P_0^T$.

The investigation of the chaotic state of motion using Poincaré mappings is also called the method of Poincaré sections. The monographs [3-7] deal with this theme. However, the construction of a Poincaré mapping itself is a complex computational problem. It is therefore usual to find PP numerically, and analytical methods for calculating PP are not used in hydrodynamic systems (with the exception of very simple, artificial examples [8]).

The following theoretical results are useful in the qualitative investigation of systems with one degree of freedom with a Hamiltonian of the standard form

$$H = \varepsilon H_1 + \varepsilon^2 H_2 + \varepsilon^3 H_3 + \dots \quad (1.2)$$

where ε is a small parameter. It is well known that an autonomous Hamilton system (a Hamiltonian of which is explicitly independent of time) is integrable. Lagrangian particles lie on one-dimensional streamlines $H(X_n, Y_n) = \text{const}$, and the motion is ordered. In the case of a non-autonomous system of standard form, the asymptotic procedure of the method of averaging in [9, 10] enables one to construct a canonical replacement of variables $X, Y \rightarrow X, Y$ which is close to an identical replacement for any integral $k > 0$ such that the equations for the new variables have the form of an autonomous Hamilton system with Hamiltonian $\bar{H}(X, Y)$, with an accuracy up to small orders of ε^{k+1} .

The Hamiltonian of the system can be reduced to the almost autonomous form $\bar{H}(X, Y) + \rho(t, X, Y)$ by a sequence of canonical substitutions. The theorem due to Neishtadt [11] establishes the existence of an unimprovable estimate $|\rho| < C_1 \exp(-C/\varepsilon)$ for the case of non-integrable systems with an analytic Hamiltonian. When $\rho = 0$, the system has an integral and, in the case of one degree of freedom, it will not be chaotic. It is precisely the exponentially small quantity $\rho(t, X, Y)$, which cannot be determined by methods of averaging, which causes chaos. The use of the method of averaging for a constructive description of the transition to chaotic motions is therefore found to be useless in system with one degree of freedom. However, the following typical pattern of the buildup of chaos as the parameter ε increases can be qualitatively ascertained. For sufficiently small ε , the sets of PP lie on the invariant curves $\bar{H}(X_n, Y_n) = \text{const}$ which are determined by methods of averaging with an accuracy up to $\rho = C_1 \exp(-C/\varepsilon)$. In this case, chaos is barely noticeable, by virtue of the smallness of ρ . For a sufficient increase in ε , the exponential contribution starts to manifest itself, the chaos becomes noticeable and the area of chaotization increases quite rapidly as ε becomes larger.

The origin of the chaos is usually associated with the existence of unstable fixed points of a Poincaré mapping [3-7]. A periodic solution with period T (with period kT) corresponds to a fixed point

$P(\mathbf{R}) = \mathbf{R}$ (the point $P^k(\mathbf{R}) = \mathbf{R}$). The problem of the Lyapunov stability of the periodic solution reduces to solving the problem of the stability of a fixed point of the mapping. The method of Lyapunov exponents [5] is used to solve it. Since the analytic form of the mapping P is unknown, the Lyapunov exponents are determined numerically. It is shown below how the mapping P and the Lyapunov exponents can be found analytically in the form of an expansion in ϵ , and in an actual hydrodynamic system with an accuracy up to ϵ^5 .

It is an exceedingly difficult problem, even for a system with one degree of freedom, to prove that there is a state of chaos using any rigorous definition of chaos. Proofs of a state of chaos for certain simple mappings have been presented in [8]. A theorem due to Mel'nikov [5], which involves evaluating a fairly complex integral, is used to prove that there is a state of chaos in the case of Hamilton systems. A state of chaos of the motions of a mathematical pendulum with a vibrating suspension point can be proved analytically by Mel'nikov's method. In such a system, with a Hamiltonian which depends periodically on time, there is a separatrix, represented in a simple analytical form, which also enables this theorem to be used and enables the uncoupling of the separatrices to be established [2]. The verification of a state of chaos, using Mel'nikov's theorem or some other method, is usually only established by means of rather lengthy numerical methods.

Several quite simple examples of the investigation of hydrodynamic systems using the numerical determination of the PP have been presented. There have been practically no investigations of the motion of a viscous fluid in a domain with a boundary which varies with time. One of these investigations [12] involved analysing the motion of particles of incompressible media with a different rheology in the thin deformed layer; conditions of zero tangential velocity were imposed on the boundaries of the layer. Numerical calculations showed that, in such a system, there was practically no chaos at low Reynolds numbers.

An efficient parametric method for constructing mappings in the phase flow of a Hamilton system is proposed below. The advantages of the parametric method over the well-known method of generating functions are established in this case and demonstrated using examples. The asymptotic theory for describing the transition to chaotic motion, which has been developed, is used to analyse the more-general motion of a highly viscous fluid in a thin deformed layer, taking account of the tangential velocity on the boundary. An explanation of the numerical experiments, using the absence of chaos or zero tangential velocity on the boundaries, is given on the basis of the Poincaré mapping which has been found. In the case of a tangential velocity on the boundary, chaos is possible at quite small amplitudes of deformation of the wave.

2. EQUATIONS DEFINING A POINCARÉ MAPPING

The *method of generating functions* is used in the case of canonical transformations [1, 2]. The same method can be used to construct Poincaré mappings. For simplicity, we will consider the Cauchy problem (1.1) for a system with one degree of freedom (although, it is easy to extend all the results to the case of a system with n degrees of freedom). The mapping $X_0, Y_0 \rightarrow X, Y$, which conserves the phase space, is represented in terms of a differentiable function of mixed variables of the form $X_0Y + S(t, X_0, Y)$ (the generating function) in the form of the relation

$$X = X_0 + \partial S / \partial Y, \quad Y_0 = Y + \partial S / \partial X_0 \quad (2.1)$$

When the condition

$$1 + \partial^2 S / \partial X_0 \partial Y > 0 \quad (2.2)$$

is satisfied, the system can be solved for X and Y which, as a result, gives a representation of a mapping with a Jacobian that is equal to unity for any function S . If the generating function S is determined from the Hamilton–Jacobi equation

$$S_t(t, X_0, Y) = H(t, X_0 + S_Y(t, X_0, Y), Y), \quad S(0, X_0, Y) = 0 \quad (2.3)$$

then the mapping (2.1) will be a solution of the Hamilton system (1.1).

In the case of a system of the standard form (1.2), the function S is represented by a series in powers of ϵ . Any finite number of terms of the expansion will define a mapping for which the phase space will be exactly conserved. The convenience of the use of the method of generating functions lies in this fact. However, there are serious drawbacks in this method. We shall indicate two of them (see the example of a harmonic oscillator below).

1. Mappings of the type (2.10) are not universal (for example, a rotation through 90° is not representable in the form of (2.1); for such a mapping, another pair of variables has to be chosen in the generating function, but it is then impossible to represent the identity transformations in these variables [1, 2]).

2. The solvability condition considerably restricts the range of variation of the parameter ε . In addition, condition (2.2) is not invariant under a transformation of a rotation of the Cartesian coordinates in the phase plane [2].

A new scheme for the parametric representation of a mapping is proposed which is free from these drawbacks. A similar, but non-parametric, method for constructing a generating function was proposed previously [2, p. 342].

The parametric method. We shall seek a mapping $\mathbf{R}_0 \rightarrow \mathbf{R}$, with a Jacobian equal to unity, in the parametric form $\mathbf{R} = \mathbf{R}(t, \mathbf{r})$, $\mathbf{R}_0 = \mathbf{R}(t, \mathbf{r})$, where \mathbf{r} , \mathbf{R}_0 and \mathbf{R} are two-dimensional vectors, $\mathbf{R}_0 = \mathbf{R}(0)$ is the starting point of the trajectory, \mathbf{R} is the point of the trajectory at the instant of time t and \mathbf{r} is a certain vector-parameter. The general form of this mapping is

$$\mathbf{R}_0 = \mathbf{r} - \frac{1}{2} I \frac{\partial \Psi}{\partial \mathbf{r}}, \quad \mathbf{R} = \mathbf{r} + \frac{1}{2} I \frac{\partial \Psi}{\partial \mathbf{r}}, \quad \frac{\partial \Psi}{\partial \mathbf{r}} = \begin{pmatrix} \Psi_x \\ \Psi_y \end{pmatrix} \quad (2.4)$$

where

$$\mathbf{R} = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad I = \begin{pmatrix} 0E \\ -E0 \end{pmatrix} \quad (2.5)$$

(I is a symplectic matrix and E is the identity matrix).

The mapping $\mathbf{R}_0 \rightarrow \mathbf{R}$ is obtained if the first equation of (2.4) is solved for the parameter \mathbf{r} and the relation $\mathbf{r}(\mathbf{R}_0)$ is then substituted into the second equation. The Jacobians of the two mappings (2.4) are equal to the one and the same function

$$J(t, \mathbf{r}) = \det \left(\frac{\partial \mathbf{R}}{\partial \mathbf{r}} \right) = \det \left(\frac{\partial \mathbf{R}_0}{\partial \mathbf{r}} \right) = 1 + \frac{1}{4} \det \left(\frac{\partial^2 \Psi}{\partial \mathbf{r} \partial \mathbf{r}} \right) \quad (2.6)$$

The Jacobian of the superposition of the mappings $\mathbf{R}_0 \rightarrow \mathbf{r} \rightarrow \mathbf{R}$ is therefore identically equal to unity. The mapping (2.4) is an analogue of (2.1) and condition (2.2) is replaced by the condition $J > 0$.

In order that the mapping (2.4) should represent the solution of the Cauchy problem (1.1) in the case of Hamilton's equations, the function Ψ must satisfy an analogue of the Hamilton-Jacobi equation (2.3)

$$\frac{\partial \Psi}{\partial t} = H \left(t, \mathbf{r} + \frac{1}{2} I \frac{\partial \Psi}{\partial \mathbf{r}} \right), \quad \Psi(0, \mathbf{r}) = 0 \quad (2.7)$$

To be specific, it is henceforth assumed that $t_0 = 0$.

For a system with one degree of freedom in coordinate form, the mapping (2.4) has the form

$$X_0 = x - \frac{1}{2} \Psi_y, \quad Y_0 = y + \frac{1}{2} \Psi_x; \quad X = x + \frac{1}{2} \Psi_y, \quad Y = y - \frac{1}{2} \Psi_x \quad (2.8)$$

This parametric form of a mapping with a Jacobian equal to unity was derived in [13] with a reference to the work of Scheffers.

Correspondingly, Eq. (2.7) for the function $\Psi(t, x, y)$ is written as

$$\frac{\partial \Psi}{\partial t} = H \left(t, x + \frac{1}{2} \frac{\partial \Psi}{\partial y}, y - \frac{1}{2} \frac{\partial \Psi}{\partial x} \right), \quad \Psi(0, x, y) = 0 \quad (2.9)$$

Formulae (2.8) and (2.9) are easily extended to systems with n degrees of freedom. In order to do this, it is sufficient to replace the parameters x and y in them with the vector-parameters \mathbf{x} and \mathbf{y} . These results are the basis of the asymptotic method for obtaining Poincaré mappings. We will now formulate them in the form of a theorem.

Theorem 1. The mapping (2.4) is a solution of problem (1.1) if and only if the function Ψ is a solution of problem (2.7).

Theorem 1 is a special case of a more general theorem on the parametrization of canonical transformations, which we shall prove below.

3. THE PARAMETRIC FORM OF CANONICAL TRANSFORMATIONS

The following theorem holds for the more general result of the parametrization of the canonical replacement of variables in Hamilton systems.

Theorem 2. Suppose the transformation of the variables is written in the parametric form

$$\begin{cases} \mathbf{q} = \mathbf{x} - \frac{1}{2} \Psi_{\mathbf{y}} \\ \mathbf{p} = \mathbf{y} + \frac{1}{2} \Psi_{\mathbf{x}} \end{cases}, \quad \begin{cases} \mathbf{Q} = \mathbf{x} + \frac{1}{2} \Psi_{\mathbf{y}} \\ \mathbf{P} = \mathbf{y} - \frac{1}{2} \Psi_{\mathbf{x}} \end{cases} \quad (3.1)$$

Then

(1) the Jacobians of the two transformations $\mathbf{q} = \mathbf{q}(t, \mathbf{x}, \mathbf{y})$, $\mathbf{p} = \mathbf{p}(t, \mathbf{x}, \mathbf{y})$ and $\mathbf{Q} = \mathbf{Q}(t, \mathbf{x}, \mathbf{y})$, $\mathbf{P} = \mathbf{P}(t, \mathbf{x}, \mathbf{y})$ are identically equal:

$$\frac{\partial(\mathbf{q}, \mathbf{p})}{\partial(\mathbf{x}, \mathbf{y})} = \frac{\partial(\mathbf{Q}, \mathbf{P})}{\partial(\mathbf{x}, \mathbf{y})} = J(t, \mathbf{x}, \mathbf{y}) \quad (3.2)$$

(2) when $J > 0$, transformation (3.1) of the variables $\mathbf{q}, \mathbf{p} \rightarrow \mathbf{Q}, \mathbf{P}$ will be canonical, translating the Hamilton system $\dot{H} = \dot{H}(t, \mathbf{q}, \mathbf{p})$ into the Hamilton system $\dot{H} = H(t, \mathbf{Q}, \mathbf{P})$, if the function Ψ is determined from the equation

$$\Psi_t(t, \mathbf{x}, \mathbf{y}) + \dot{H}(t, \mathbf{q}, \mathbf{p}) = H(t, \mathbf{Q}, \mathbf{P}) \quad (3.3)$$

where the arguments \mathbf{q}, \mathbf{p} and \mathbf{Q}, \mathbf{P} in the Hamiltonians H and \dot{H} are expressed in terms of the parameters \mathbf{x}, \mathbf{y} in accordance with (3.1).

We will now prove Assertion 2 using the canonical property criterion, according to which the differential form $\delta F = \mathbf{P} \delta \mathbf{Q} - \mathbf{p} \delta \mathbf{q} - (H - \dot{H}) \delta t$ is a complete differential of a certain function $\delta F(t, \mathbf{x}, \mathbf{y}) = F_t \delta t + F_x \delta \mathbf{x} + F_y \delta \mathbf{y}$ [1].

Instead of $\mathbf{q}, \mathbf{p}, \mathbf{Q}, \mathbf{P}$, we substitute their expressions (3.1) in terms of the parameters \mathbf{x} and \mathbf{y} into the differential form δF and replace $H - \dot{H}$ in terms of Ψ_t in accordance with Eq. (3.3). After obvious algebra, we obtain

$$\begin{aligned} \delta F &= (\mathbf{y} - \frac{1}{2} \Psi_{\mathbf{x}})(\delta \mathbf{x} + \frac{1}{2} \Psi_{\mathbf{y}t} \delta t + \frac{1}{2} \Psi_{\mathbf{y}x} \delta \mathbf{x} + \frac{1}{2} \Psi_{\mathbf{y}y} \delta \mathbf{y}) - \\ &- (\mathbf{y} + \frac{1}{2} \Psi_{\mathbf{x}})(\delta \mathbf{x} - \frac{1}{2} \Psi_{\mathbf{y}t} \delta t - \frac{1}{2} \Psi_{\mathbf{y}x} \delta \mathbf{x} - \frac{1}{2} \Psi_{\mathbf{y}y} \delta \mathbf{y}) - \Psi_t \delta t = \\ &= \mathbf{y}(\Psi_{\mathbf{y}t} \delta t + \Psi_{\mathbf{y}x} \delta \mathbf{x} + \Psi_{\mathbf{y}y} \delta \mathbf{y}) - \Psi_x \delta \mathbf{x} - \Psi_t \delta t = \delta(\mathbf{y} \Psi_{\mathbf{y}} - \Psi) \end{aligned}$$

which it was required to prove.

Assertion 1 follows from the fact that, in the case of a canonical substitution, the Jacobian is identically equal to unity $\partial(\mathbf{Q}, \mathbf{P})/\partial(\mathbf{q}, \mathbf{p}) = 1$.

Here, $\mathbf{q}, \mathbf{p}, \mathbf{Q}, \mathbf{P}, \mathbf{x}, \mathbf{y}$ are understood as being n -dimensional vectors. Hence, the method of canonical substitutions which has been described refers to Hamilton systems of arbitrary order n .

In the special case when $\dot{H} = 0$, the variables \mathbf{q} and \mathbf{p} are transformed into the initial point $\mathbf{q} = \mathbf{Q}(0)$, $\mathbf{p} = \mathbf{P}(0)$ of the trajectory $\mathbf{Q} = \mathbf{Q}(t)$, $\mathbf{P} = \mathbf{P}(t)$ of a system with Hamiltonian $H(t, \mathbf{Q}, \mathbf{P})$. Hence, Theorem 1 is also proved.

4. AN ASYMPTOTIC METHOD FOR DETERMINING A POINCARÉ MAPPING

Suppose the Hamiltonian of a system of standard form is represented in the form of the power series

(1.2). The solution of Eq. (2.7) with a null initial condition is then represented in the form of a series in powers of ϵ

$$\Psi = \epsilon\Psi_1 + \epsilon^2\Psi_2 + \epsilon^3\Psi_3 + \dots \tag{4.1}$$

In order to calculate the coefficients of this series, we substitute expansion (4.1) into the right-hand side of Eq. (2.7) and expand it in a series in powers of ϵ

$$H(t, \mathbf{x} + \frac{1}{2}\Psi_y, \mathbf{y} - \frac{1}{2}\Psi_x, \epsilon) = \epsilon H_1 + \epsilon^2 [H_2 - \frac{1}{2}\{H_1, \Psi_1\}] + \epsilon^3 [H_3 - \frac{1}{2}(\{H_2, \Psi_1\} + \{H_1, \Psi_2\}) + \frac{1}{8}(H_{1xx}\Psi_{1y}\Psi_{1y} - 2H_{1xy}\Psi_{1x}\Psi_{1y} + H_{1yy}\Psi_{1x}\Psi_{1x})] + \dots$$

The derivatives of Ψ_n are then determined as the coefficients of the powers ϵ^n

$$\begin{aligned} \Psi_{1t} &= H_1, \quad \Psi_{2t} = H_2 - \frac{1}{2}\{H_1, \Psi_1\} \\ \Psi_{3t} &= H_3 - \frac{1}{2}(\{H_2, \Psi_1\} + \{H_1, \Psi_2\}) + \frac{1}{8}(H_{1xx}\Psi_{1y}\Psi_{1y} - 2H_{1xy}\Psi_{1x}\Psi_{1y} + H_{1yy}\Psi_{1x}\Psi_{1x}), \dots \end{aligned} \tag{4.2}$$

Hence, by integration with respect to t , taking account of the conditions $\Psi_n = 0$ when $t = 0$, the coefficients of series (4.1) are expressed in terms of the specified coefficients of series (1.2). In the case of a Poincaré mapping after a period T , we obtain, up to third-order infinitesimals

$$\Psi = \int_0^T \left[\epsilon H_1(t, \mathbf{x}, \mathbf{y}) + \epsilon^2 \left(H_2(t, \mathbf{x}, \mathbf{y}) - \frac{1}{2} \left\{ H_1(t, \mathbf{x}, \mathbf{y}), \int_0^t H_1(t', \mathbf{x}, \mathbf{y}) dt' \right\} \right) \right] dt \tag{4.3}$$

($\{f, h\} = f_y h_x - f_x h_y$ is a Poisson bracket).

Formula (4.3) can be written more briefly in the form

$$\Psi = \int_0^T \left[H(t, \mathbf{x}, \mathbf{y}) - \frac{1}{2} \left\{ H(t, \mathbf{x}, \mathbf{y}), \int_0^t H(t', \mathbf{x}, \mathbf{y}) dt' \right\} \right] dt + O(\epsilon^3)$$

The convergence of series (4.1) is proved by the majorant method in the same way as for a general system of differential equations of standard form [14]. A Poincaré mapping is therefore an analytic function with respect to the parameter ϵ . This is also in accord with a known theorem concerning the analyticity of the solution of a differential equation in a parameter.

5. A POINCARÉ MAPPING FOR AUTONOMOUS HAMILTON SYSTEMS

The procedure for expanding the function Ψ in a power series is simplified considerably in the case of an autonomous Hamilton system with Hamiltonian $H(\mathbf{X}, \mathbf{Y})$.

It can be shown that the solution of Eq. (2.7) is an odd function with respect to the argument t .

Actually, the integral $H(\mathbf{R}(t)) = H(\mathbf{R}_0)$ holds in the case of an autonomous Hamilton system. Suppose $\Psi(t, \mathbf{x}, \mathbf{y})$ is the solution of (2.7). Then, according to the theorem proved above

$$\mathbf{R}(t) = \mathbf{r} + \frac{1}{2}I \frac{\partial \Psi}{\partial \mathbf{r}}, \quad \mathbf{R}_0 = \mathbf{r} - \frac{1}{2}I \frac{\partial \Psi}{\partial \mathbf{r}}, \quad H\left(\mathbf{r} + \frac{1}{2}I \frac{\partial \Psi}{\partial \mathbf{r}}\right) = H\left(\mathbf{r} - \frac{1}{2}I \frac{\partial \Psi}{\partial \mathbf{r}}\right)$$

for any real t . Next, on making the substitution $t = -\tau$, $\Psi = -\tilde{\Psi}$ in Eq. (2.7), we have

$$\frac{\partial \tilde{\Psi}}{\partial \tau} = H\left(\mathbf{r} - \frac{1}{2}I \frac{\partial \tilde{\Psi}}{\partial \mathbf{r}}\right) = H\left(\mathbf{r} + \frac{1}{2}I \frac{\partial \tilde{\Psi}}{\partial \mathbf{r}}\right), \quad \tilde{\Psi}(0, \mathbf{r}) = 0$$

The resulting boundary-value problem is identical to the initial problem (2.7). By virtue of uniqueness, we conclude that $\Psi(t, \mathbf{x}, \mathbf{y}) = -\Psi(-t, \mathbf{x}, \mathbf{y})$.

The property which has been established can be used to simplify the calculation of the series for the function Ψ in the case of a Hamiltonian of standard form (1.2). To do this, we formally write the Hamiltonian in the form

$$H = \varepsilon \tilde{H}(\mathbf{X}, \mathbf{Y}, \varepsilon_1), \quad \tilde{H} = H_1 + \varepsilon_1 H_2 + \varepsilon_1^2 H_3 + \dots$$

When $\varepsilon_1 = \varepsilon$, this Hamiltonian is identical to the initial Hamiltonian. According to the property which has been proved, the solution of (2.7), when $\varepsilon_1 = \text{const}$, can be represented by a series in odd powers of εt

$$\Psi = (\varepsilon t) \tilde{\Psi}_1 + (\varepsilon t)^3 \tilde{\Psi}_3 + (\varepsilon t)^5 \tilde{\Psi}_5 + \dots \tag{5.1}$$

The coefficients of this series $\tilde{\Psi}_n(\mathbf{x}, \mathbf{y}, \varepsilon_1)$ are calculated from Eq. (2.7) much more simply than in the non-autonomous case. Thus, in order to obtain the expansion to terms of the order of $(\varepsilon t)^5$ infinitesimals, it is only necessary to calculate two coefficients and not the four coefficients required in the general case, that is

$$\Psi = (\varepsilon t) \tilde{H}(\mathbf{x}, \mathbf{y}, \varepsilon_1) + \frac{1}{8} (\varepsilon t)^3 [\tilde{H}_{xx} \tilde{H}_y \tilde{H}_y - 2 \tilde{H}_{xy} \tilde{H}_x \tilde{H}_y + \tilde{H}_{yy} \tilde{H}_x \tilde{H}_x] + O(\varepsilon t)^5 \tag{5.2}$$

In order to obtain the final expansion in powers of ε , it is necessary to put $\varepsilon = \varepsilon_1$ and to substitute the series

$$\tilde{H}(\mathbf{X}, \mathbf{Y}, \varepsilon) = H_1 + \varepsilon H_2 + \varepsilon^2 H_3 + \dots$$

into expression (5.2).

Since a non-autonomous Hamilton system can be reduced to an autonomous Hamilton system with an increase in the number of degrees of freedom of one [1], the procedure for obtaining the Poincaré mapping in the case of a general non-autonomous system can be reduced to the procedure which has been described for an autonomous system. The considerable simplifications in the case of an autonomous system can be used to obtain a power series with a large number of terms with fewer operations. Thus, by using an analogue of formula (5.2), we obtain a mapping which is more accurate by two orders of magnitude than formula (4.3).

6. THE AVERAGING PROCEDURE

The averaging procedure using generating functions has been described earlier [9]. For a Hamiltonian which is periodic with respect to time the parametric method enables one to construct an averaged Hamiltonian no less effectively. Examples of such calculations up to terms of the order of ε^3 have been given in [15, 12, 16].

We will now present a general procedure for constructing the asymptotic series of the averaged Hamiltonian

$$\bar{H}(\mathbf{X}, \mathbf{Y}, \varepsilon) = \varepsilon \bar{H}_1 + \varepsilon^2 \bar{H}_2 + \varepsilon^3 \bar{H}_3 + \dots \tag{6.1}$$

using the parametric method.

Suppose the initial Hamiltonian is periodic with respect to time with period T and that it is represented by series (1.2) with known coefficients. Then, the coefficients of series (4.1) for the function $\Psi(t, \mathbf{x}, \mathbf{y})$ can be successively calculated using formulae (4.2). We shall therefore assume that series (4.1) is known.

Next, it is necessary to solve the following equation

$$\Psi_t = \bar{H} \left(\mathbf{x} + \frac{1}{2} \bar{\Psi}_y, \mathbf{y} - \frac{1}{2} \bar{\Psi}_x, \varepsilon \right), \quad \bar{\Psi}(0, \mathbf{x}, \mathbf{y}, \varepsilon) = 0 \tag{6.2}$$

$$\Psi(T, \mathbf{x}, \mathbf{y}, \varepsilon) - \bar{\Psi}(T, \mathbf{x}, \mathbf{y}, \varepsilon) = O(\varepsilon^{k+1}) \tag{6.3}$$

where $\Psi(t, \mathbf{x}, \mathbf{y}, \varepsilon)$ is the Poincaré mapping of the initial system and $\bar{\Psi}(t, \mathbf{x}, \mathbf{y}, \varepsilon)$ is the Poincaré mapping corresponding to a system with Hamiltonian $\bar{H}(\mathbf{x}, \mathbf{y}, \varepsilon)$. Equation (6.2) connects the mapping function

$\bar{\Psi}(t, \mathbf{x}, \mathbf{y}, \varepsilon)$ and the Hamiltonian \bar{H} generating this mapping. Equation (6.3) means that the locations of the PP of the initial and averaged systems are identical up to terms of the order of ε^{k+1} after a period.

Taking account of (6.1), we can write the right-hand side of Eq. (6.2) in the form

$$\bar{H}(\mathbf{X}, \mathbf{Y}, \varepsilon) = \varepsilon \bar{H}(\mathbf{X}, \mathbf{Y}, \varepsilon_1), \quad \bar{H} = \bar{H}_1 + \varepsilon_1 \bar{H}_2 + \varepsilon_1^2 \bar{H}_3 + \dots$$

Then, according to expansions (5.1) and (5.2), the solution of Eq. (6.2) will have the form

$$\bar{\Psi}(t, \mathbf{x}, \mathbf{y}, \varepsilon) = (\varepsilon t) \bar{\Psi}_1 + (\varepsilon t)^3 \bar{\Psi}_3 + (\varepsilon t)^5 \bar{\Psi}_5 + \dots \tag{6.4}$$

$$\bar{\Psi}_1 = \bar{H}(\mathbf{x}, \mathbf{y}, \varepsilon_1), \quad \bar{\Psi}_3 = \frac{1}{8} (\bar{H}_{xx} \bar{H}_y \bar{H}_y - 2 \bar{H}_{xy} \bar{H}_x \bar{H}_y + \bar{H}_{yy} \bar{H}_x \bar{H}_x), \dots$$

We will find the coefficients of the asymptotic series of the averaged Hamiltonian (6.1) from equality (6.3), with an accuracy specified in advance. For example, the averaged Hamiltonian, multiplied by the period T , and the mapping function are identical up to terms of the order of ε^3

$$T \bar{H}(\mathbf{X}, \mathbf{Y}, \varepsilon) = \Psi(T, \mathbf{X}, \mathbf{Y}, \varepsilon) \tag{6.5}$$

(There is no such simple relation in this approximation in the case of a generating function.)

On taking account of expression (4.3), we obtain the well-known formula for the averaged Hamiltonian in this approximation from equality (6.5) [9]

$$\bar{H} = \frac{1}{T} \bar{\Psi} = \left\langle H - \frac{1}{2} \left\{ H, \int_0^t H dt \right\} \right\rangle + O(\varepsilon^3) \tag{6.6}$$

where the angular brackets denote averaging over the period.

There will be similar simplifications in the higher approximations. The coefficients of the series of the averaged Hamiltonian are obtained up to terms of the order of ε^5 from the equalities $\Psi(T, \mathbf{x}, \mathbf{y}) = (\varepsilon T) \bar{\Psi}_1 + (\varepsilon T)^3 \bar{\Psi}_3$ and (6.4). Hence, it is easy to obtain the coefficients in this approximation

$$\begin{aligned} \bar{H}_1 &= \frac{1}{T} \Psi_1, \quad \bar{H}_2 = \frac{1}{T} \Psi_2 \\ \bar{H}_3 &= \frac{1}{T} \left[\Psi_3 - \frac{1}{8} T^3 (\bar{\Psi}_{1xx} \bar{\Psi}_{1y} \Psi_{1y} - 2 \bar{\Psi}_{1xy} \bar{\Psi}_{1x} \bar{\Psi}_{1y} + \bar{\Psi}_{1yy} \bar{\Psi}_{1x} \bar{\Psi}_{1x}) \right] + \dots \end{aligned} \tag{6.7}$$

The parametric method is therefore more effective than the method of generating functions [9] for calculating the coefficients of the asymptotic expansions of an averaged Hamiltonian.

7. THE MAPPING OF A SMALL DOMAIN

We shall assume that the Hamiltonian $H(t, \mathbf{X}, \mathbf{Y}, \varepsilon)$ is a continuously, doubly differentiable function of the coordinates \mathbf{X} and \mathbf{Y} . As a function of time, the Hamiltonian has a period T . The coordinates $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ vary in a certain compact domain.

According to the theory of finite deformations [17], the matrix $A = \partial \mathbf{R} / \partial \mathbf{R}_0$ ($\mathbf{R} = (\mathbf{X}, \mathbf{Y})$) defines the mapping of a small neighbourhood of the point $\mathbf{R}_0 = (\mathbf{X}_0, \mathbf{Y}_0)$. Using the mapping (2.7), the matrix A can be expressed in terms of the derivatives of the mapping function $\Psi = \Psi(T, \mathbf{x}, \mathbf{y}, \varepsilon)$.

According to representation (2.4), the matrix A is equal to the product of the inverse of the mapping matrix $\mathbf{r} \rightarrow \mathbf{R}_0$ and the mapping matrix $\mathbf{r} \rightarrow \mathbf{R}$ ($\mathbf{r} = (\mathbf{x}, \mathbf{y})$)

$$\begin{aligned} A &= A_-^{-1} A_+; \quad A_{\pm} = E \pm \frac{1}{2} J \frac{\partial^2 \Psi}{\partial \mathbf{R}^2} \\ \frac{\partial^2 \Psi}{\partial \mathbf{R}^2} &= \begin{vmatrix} \Psi_{xx} & \Psi_{xy} \\ \Psi_{xy} & \Psi_{yy} \end{vmatrix}, \quad J \frac{\partial^2 \Psi}{\partial \mathbf{R}^2} = \begin{vmatrix} \Psi_{xy} & \Psi_{yy} \\ -\Psi_{xx} & -\Psi_{xy} \end{vmatrix} \end{aligned}$$

Using the identity $A_-^{-1} = A_+ / J$, the matrix A can be represented in the form

$$A = \frac{1}{J} \left[(2 - J)E + \begin{vmatrix} \Psi_{xy} & \Psi_{yy} \\ -\Psi_{xx} & -\Psi_{xy} \end{vmatrix} \right] \tag{7.1}$$

where J is the Jacobian defined by formula (2.6).

We will now carry out a further study of the local mapping for two-dimensional domains $(X_0, Y_0) \rightarrow (X, Y)$. We will express the components of the local mapping matrix in terms of the second derivatives of the function Ψ and, for comparison, in terms of the second derivatives of the generating function, representing this mapping in the form of (2.1)

$$\begin{aligned} A_{11} &= \frac{2 - J}{J} + \frac{\Psi_{xy}}{J} = \frac{(1 + S_{XY})^2 - S_{XX}S_{YY}}{1 + S_{XY}} \\ A_{12} &= \frac{\Psi_{yy}}{J} = \frac{S_{YY}}{1 + S_{XY}}, \quad A_{21} = -\frac{\Psi_{xx}}{J} = -\frac{S_{XX}}{1 + S_{XY}} \\ A_{22} &= \frac{2 - J}{J} - \frac{\Psi_{xy}}{J} = \frac{1}{1 + S_{XY}} \end{aligned} \tag{7.2}$$

The mapping of a small neighbourhood $(\delta X_0, \delta Y_0) \rightarrow (\delta X, \delta Y)$ with a matrix A has two independent invariants with respect to rotations of the system of coordinates

$$I_1 = A_{11} + A_{22}, \quad I_2 = A_{12} - A_{21}$$

In the case of the parametric mapping J and $\Delta\Psi = \Psi_{xx} + \Psi_{yy}$ are also invariants since they are expressed in terms of the invariants I_1 and I_2

$$J = \frac{4}{I_1 + 2}, \quad \Delta\Psi = \frac{4I_2}{I_1 + 2}$$

Hence, the condition $J > 0$ for the existence of a parametric mapping is invariant and it can be represented in the form $I_1 + 2 > 0$.

We will now show that the condition for the existence of a mapping (2.2) with an arbitrary function $1 + S_{XY} > 0$ is not invariant. In fact, as a consequence of the last relation of (7.2), this condition can be represented in the form $A_{22} > 0$. The satisfaction of this condition depends on the choice of the system of coordinates. In the system of coordinates X', Y' , rotated by an angle θ , this condition is

$$\frac{1}{2}(A_{11} + A_{22}) - \frac{1}{2}(A_{11} - A_{22})\cos 2\theta + \frac{1}{2}(A_{12} + A_{21})\sin 2\theta > 0$$

There will always be a dependence on the angle θ , apart from the case when $(A_{11} - A_{22})^2 + (A_{12} - A_{21})^2 = 0$ or $I_1^2 + I_2^2 = 4$.

The invariance of the condition for the existence of a parametric mapping is the first important advantage over a classical mapping with a generating function.

If the parametric mapping and mapping (2.1) are calculated with the same asymptotic accuracy, the domain of existence of the parametric mapping is much wider than the domain of existence of mapping (2.1). This second advantage of the parametric mapping will be demonstrated below using real examples.

The characteristic factor m , which determines the nature of the stability of a fixed point, is a root of the characteristic polynomial $m^2 - I_1 m + 1 = 0$ and depends solely on I_1 or $J(I_1)$. It is well known that the instability of a fixed point is one of the important conditions for a transition to chaos. The invariant I_1 and $J(I_1)$, which depends on it, therefore plays a fundamental role in the investigation of the transition to a state of chaos in the case of two-dimensional mappings.

We will now distinguish the degenerate case

$$\langle H_1 \rangle = \frac{1}{T} \int_0^T H_1 dt = 0$$

from a system of the standard form (1.2). In this case, formula (4.3), written up to terms in ϵ^3 , is simplified

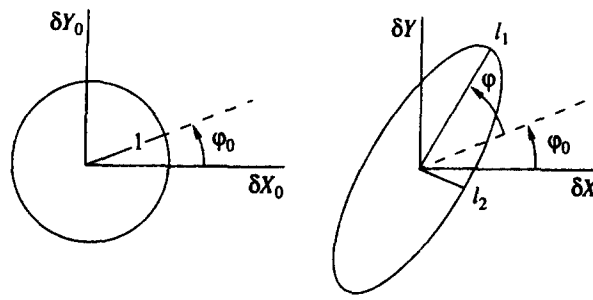


Fig. 1

$$\Psi = \epsilon^2 \int_0^T \left(H_2 - \frac{1}{2} \left\{ H_1, \int_0^t H_1 dt' \right\} \right) dt + O(\epsilon^3)$$

and the invariant J is determined using formula (2.6) up to terms of the fifth order in ϵ

$$J = 1 + \frac{1}{4}(\Psi_{xx}\Psi_{yy} - \Psi_{xy}^2) + O(\epsilon^5)$$

Consequently, m is also determined with this accuracy. In this case, the parametric mapping is calculated more simply and, moreover, it enables one to investigate the stability of a fixed point of the mapping with a higher accuracy and to describe the transition to a chaotic state, as is shown below in the hydrodynamic example.

According to a theorem on polar expansion, which is well known in analytical geometry, the matrix A can always be represented in the form of the sum of a rotation matrix and a symmetric matrix. In its turn, the symmetric matrix can be brought to the principal axes by means of a rotation of the system of coordinates. Hence, we obtain

$$A = C(\varphi)C(\varphi_0) \begin{vmatrix} l_1 & 0 \\ 0 & l_2 \end{vmatrix} C(-\varphi_0), \quad l_1 l_2 = 1, \quad C(\varphi) = \begin{vmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{vmatrix} \quad (7.3)$$

The mapping $\delta X = A\delta X_0$ transforms the unit circle into an ellipse of the same area with axes l_1, l_2 . The angle φ_0 determines the direction of the filament in the circle $|\delta X_0| = 1$ which is lengthened to the greatest extent during the transformation. The angle φ defines the rotation of this filament (Fig. 1). Using relations (7.1) and (7.3), all the above-mentioned characteristics of the mapping can be expressed in terms of the elements of the Hessian matrix $\Psi_{xx}, \Psi_{xy}, \Psi_{yy}$.

The greatest and the smallest extensions of the filaments are

$$l_1 = R + \sqrt{R^2 - 1}, \quad l_2 = R - \sqrt{R^2 - 1} \quad (7.4)$$

$$R^2 = \frac{1}{4J^2} ((2 - J)^2 + (\Psi_{xx} + \Psi_{yy})^2) + \frac{1}{2}$$

We present the expressions for the angles φ and φ_0

$$\text{tg } \varphi = \frac{A_{21} - A_{12}}{A_{11} + A_{22}}, \quad \text{tg}(\varphi + 2\varphi_0) = \frac{A_{12} + A_{21}}{A_{11} - A_{22}}$$

The characteristics $l_1, \varphi, m_1, \lambda_1$ which have been written out above are also invariants and are expressed in terms of the two basic invariants. The angle φ_0 is an invariant and depends on the choice of the axes of the coordinates $\delta X, \delta Y$.

8. THE ADVANTAGES OF THE PARAMETRIC MAPPING

In the parametric method, the function $\Psi(t, x, y)$ plays the role of a generating function. As in the method of generating functions, $\Psi(t, x, y)$ is determined from an equation of the Hamilton–Jacobi type. Any

finite sum of the first terms of the series in ϵ for Ψ will give a mapping which exactly conserves the phase space, subject to the condition that $J > 1$.

However, a number of advantages of the parametric method over the method of generating functions was noted above. They consist of the following.

1. It is impossible to express mappings of a very simple form in terms of a generating function. For example, it is impossible to express a rotation by 90° (the mapping $X = Y_0, Y = -X_0$) in terms of $S(X_0, Y)$. This mapping can be expressed in terms of a mapping function with another pair of variables, but it is then impossible to represent the identity mapping in terms of it. In this sense, the parametric representation is universal. For the identity transformation $\Psi = 0$ and, for a rotation by 90° , $1 + \partial^2 S / \partial X_0 \partial Y > 0$.

2. The condition of solvability (2.2), generally speaking, depends on the choice of the Cartesian system of coordinates X and Y , whereas in the parametric method, the condition $J > 0$ is invariant under rotations of the X and Y axes.

3. In approximations of the same accuracy with respect to the small parameter ϵ , the range of variation of the parameter ϵ , which satisfies the condition $J > 0$, is much wider compared with the range of variation of the parameter ϵ which satisfies the condition $1 + \partial^2 S / \partial X_0 \partial Y > 0$.

4. The coefficients Ψ_n of the series $\Psi = \epsilon \Psi_1 + \epsilon^2 \Psi_2 + \dots$ are considerably smaller than the coefficients of the series S_n for the generating function.

5. In the case of an autonomous system, the function Ψ is represented by a series in odd powers of ϵ .

6. A mapping after a time t and the averaged Hamiltonian are connected, up to terms of the order of ϵ^3 , by the relation $\Psi = T\bar{H}(x, y)$.

We will now demonstrate the advantages of the parametric method in an example in which the mapping is found exactly.

The forced vibrations of an oscillator. The Hamiltonian has the form

$$H = \epsilon \left[\frac{1}{2} (X^2 + Y^2) + Xb \sin(t_0 + t) \right]$$

The system of equations is integrated exactly and the coordinates X_n, Y_n of the points of succession over a period at the instants of time $t_n = 2\pi n$ are

$$\begin{aligned} X_n - X_c &= (X_{n-1} - X_c) \cos(2\pi\epsilon) + (Y_{n-1} - Y_c) \sin(2\pi\epsilon) \\ Y_n - Y_c &= - (X_{n-1} - X_c) \sin(2\pi\epsilon) + (Y_{n-1} - Y_c) \cos(2\pi\epsilon) \\ X_c &= \frac{b\epsilon^2}{1 - \epsilon^2} \sin t_0, \quad Y_c = \frac{b\epsilon}{1 - \epsilon^2} \cos t_0 \end{aligned}$$

The points of succession lie on a circle with its centre at X_c, Y_c with an angular distance from one another of $2\pi\epsilon$.

The parametric method gives

$$\Psi(x, y) = \text{tg } \pi\epsilon [(x - X_c)^2 + (y - Y_c)^2], \quad J = 1 / \cos^2(\pi\epsilon) > 0$$

According to the method of generating functions, we have

$$\begin{aligned} S(X_0, Y) &= \frac{1 - \cos 2\pi\epsilon}{\cos 2\pi\epsilon} (X_0 - X_c)(Y - Y_c) + \frac{1}{2} \text{tg } 2\pi\epsilon [(X_0 - X_c)^2 + (Y - Y_c)^2] \\ 1 + S_{XY} &= \frac{1}{\cos 2\pi\epsilon} > 0 \end{aligned}$$

It is clear from the above example that:

- (a) the formula for Ψ is shorter than that for S ;
- (b) the radii of convergence of the series in powers of ϵ for Ψ are twice as great as the radii of convergence of the series for S (rotations of up to 180° are permitted in the parametric method but only up to 90° in the method of generating functions);
- (c) the coefficients of the series Ψ_n are approximately 2^n times less than the coefficients of S_n and, correspondingly, for the residual terms of these series r_n and R_n , we have $r_n \approx 2^{-n} R_n$.

All of advantages 1-6 are easily verified.

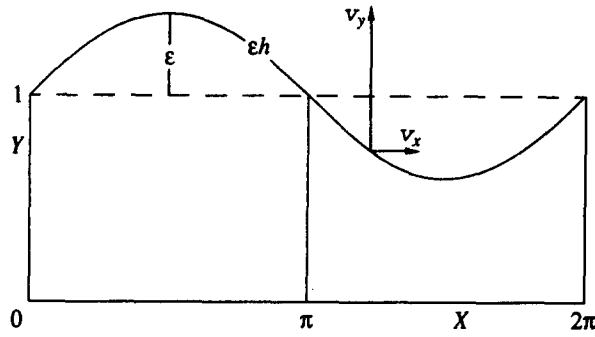


Fig. 2

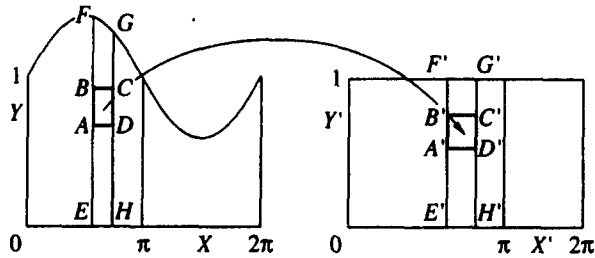


Fig. 3

The example, presented below, is a rather complex hydrodynamic system in which the transition to dynamic chaos for sufficiently high values of the parameter is successfully described by the parametric method. It is shown that, in this domain of parameters, the classical method of constructing Poincaré mappings is fundamentally inapplicable.

9. THE MOTION OF THE PARTICLES OF A VISCOUS FLUID IN A THIN LAYER

A slow plane-parallel flow of an incompressible viscous fluid in a thin layer is considered. This layer is bounded by a flat bottom $Y = 0$, rectilinear side walls $X = 0, X = 2\pi$ and an upper boundary which is deformed according to a specified periodic law $Y = 1 + \epsilon h(t, X)$. A no-slip condition is assumed on the lower boundary $Y = 0$, a non-permeability condition $v_y = \epsilon \partial h / \partial t + v_x \partial h / \partial x$ is imposed on the upper boundary, and a condition is imposed on the tangential component $v_x = 3\epsilon^2 \alpha (1 - \cos x)$ (Fig. 2). The flow rate on the side walls is equal to zero. This problem has been studied previously in [12] for $\alpha = 0$. The Hamiltonian of the system (the stream function) in the thin-layer approximation is defined in the same way as in [12].

$$\begin{aligned}
 H(t, X, Y) &= \epsilon q(3\tilde{Y}^2 - 2\tilde{Y}^3) - 3\epsilon^2 \alpha (\tilde{Y}^2 - \tilde{Y}^3)(1 - \cos X) & (9.1) \\
 \partial h / \partial t + \partial q / \partial X &= 0, \quad q(0) = q(2\pi) = 0 \\
 \tilde{Y} &= Y / (1 + \epsilon h(t, X))
 \end{aligned}$$

Here X and Y are dimensionless coordinates, the coordinate $X/(2\pi)$ is divided by the length of the layer and the coordinate Y is divided by its thickness. The dimensionless function for the flow rate $\epsilon q(t, X)$ across a section $X = \text{const}$ is expressed in terms of the function for the deformation of the layer ϵh using the equation for the conservation of mass.

The motion of the fluid particles is determined from the solution of a system of Hamilton equations with Hamiltonian (9.1). Investigation of the system when $\alpha = 0$ [12] showed that there is no transition to chaos up to $\epsilon < 0.8$.

If terms of the order of ϵ^2 of the velocity v_x on the boundary are taken into account qualitative changes occur in the motion of the fluid particles.

In order to investigate the system. It is convenient to use the canonical substitution

$$X, Y, H(t, X, Y) \rightarrow \tilde{X}, \tilde{Y}, \tilde{H}(t, \tilde{X}, \tilde{Y})$$

with generating function

$$S(t, X, \tilde{Y}) = X\tilde{Y} + \epsilon\tilde{Y} \int_0^X h(t, X) dX$$

The variables and the Hamiltonian then become

$$\begin{aligned} \tilde{X} &= \frac{\partial S}{\partial \tilde{Y}} = X + \epsilon \int_0^X h(t, X) dX, & Y &= \frac{\partial S}{\partial X} = Y(1 + \epsilon h(t, X)) \\ \tilde{H}(t, \tilde{X}, \tilde{Y}) &= H + \partial S / \partial t = H - \epsilon q \tilde{Y} \end{aligned}$$

The convenience of this substitution lies in the fact that, in the variable \tilde{X}, \tilde{Y} , the system has the Hamiltonian form and the domain of the variables is a fixed rectangle $\tilde{X} \in (0, 2\pi), \tilde{Y} \in (0, 1)$ (Fig. 3).

The system can be investigated analytically in the case of a general deformation law. The mapping function can be calculated up to terms of the order of ϵ^3 using formulae (4.3)

$$\Psi(x, y) = 6\pi\epsilon^2 [2\langle qh \rangle (2y - 1)(1 - y)^2 y^2 - \alpha(1 - y)y^2(1 - \cos x)] \tag{9.2}$$

where q is found from the equation of conservation of mass

$$\partial h / \partial t + \partial q / \partial X = 0, \quad q(0) = q(2\pi) = 0$$

The equalities

$$\int_0^T \left(h, \int_0^{\tilde{X}} h dX \right) dt = - \int_0^T \left(h \int_0^{\tilde{X}} h, dX \right) dt = \int_0^T h q dt$$

are used in deriving formula (9.2).

We shall confine ourselves to investigating a special case of deformation using the law for a travelling wave $h = \sin(X - t)$. From the equation of conservation of mass, we calculate

$$q = \sin(X - t) + \sin t, \quad \langle qh \rangle = \frac{1}{2\pi} \int_0^{2\pi} q h dt = \frac{1}{2} (1 - \cos X)$$

and, substituting this into expression (9.2), we obtain

$$\begin{aligned} \Psi(x, y) &= 6\pi\epsilon^2 F(y, \alpha)(1 - \cos X) \\ F(y, \alpha) &= (2y - 1)(1 - y)^2 y^2 - \alpha(1 - y)y^2 \\ J &= 1 + 9\pi^2\epsilon^4 [F(y, \alpha)F''(y, \alpha)\cos x(1 - \cos x) - (F'(y, \alpha)\sin x)^2] > 0 \end{aligned} \tag{9.3}$$

The variables x and y occupy a region of the same rectangle $x \in (0, 2\pi), y \in (0, 1)$. Hence, a two-parameter family of mappings is obtained for the initial system.

If $\alpha < 0.236$, the Jacobian J attains the minimum value when $\cos X = -1$, and if $\alpha > 0.236$, it attains the minimum value when $\cos X = 0$. Hence, the domain of applicability of transformation (9.3) is

$$\begin{aligned} \min J &= 1 - 9\pi^2\epsilon^4 \max(2F(y, \alpha)F''(y, \alpha)) > 0 \quad \text{when } \alpha < 0.236 \\ \min J &= 1 - 9\pi^2\epsilon^4 \max(F'(y, \alpha))^2 = 1 - 9\pi^2\epsilon^4\alpha^2 > 0 \quad \text{when } \alpha > 0.236 \end{aligned}$$

Investigation of the function $F'(y, \alpha)$ at the maximum shows that, when $\alpha \in (-\infty, -1/2) \cup (1/8, \infty)$, the function $|F'(y, \alpha)|$ attains its greatest value, equal to α , on the boundary $y = 1$. In the case of the remaining α , the function $|F'(y, \alpha)|$ attains the greatest value in the interval $y \in (0, 1)$.

The domain $\min J > 0$ in the plane of the parameters (α, ϵ) is shown in Fig. 4. The boundary is defined by the equation $\min J = \Phi(\epsilon, \alpha) = 0$ and it is a piecewise-smooth line.

For comparison, we present the asymptotic mapping formulae obtained using the method of generating functions in the same approximation as in (9.3). The mapping will be determined using

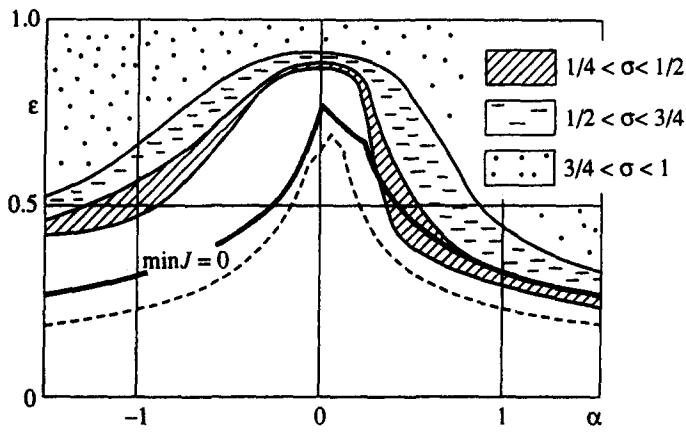


Fig. 4

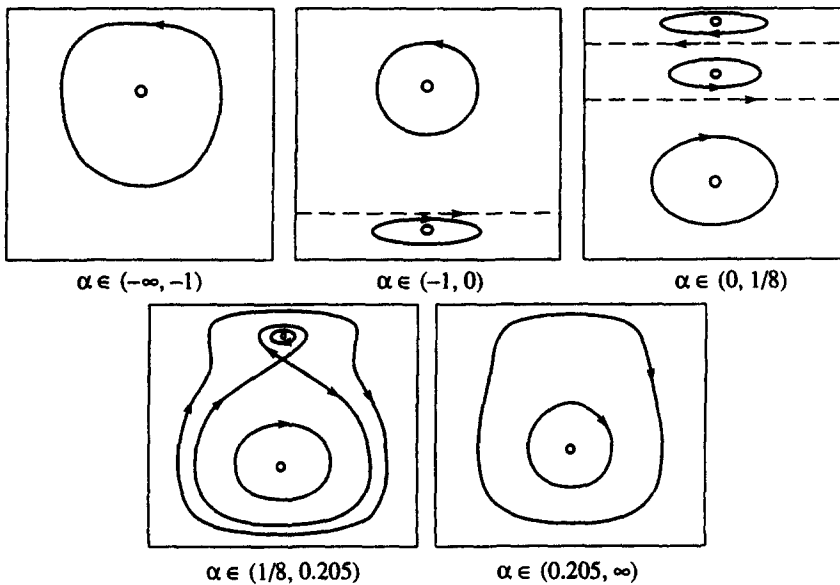


Fig. 5

formulae (2.1) with the function $S(X_0, Y) = \Psi(X_0, Y)$, where the function Ψ is the same as in (9.3). The condition for the transformation to be applicable is

$$\min(1 + \Psi_{X_0 Y}) = 1 + 6\pi\epsilon^2 \min(F'(Y, \alpha)) \sin X_0 > 0 = 1 - 6\pi\epsilon^2 \max |F'(Y, \alpha)| > 0$$

When $\alpha \in (-\infty, -1/2) \cup (1/8, \infty)$, we obtain $6\pi\epsilon^2 |\alpha| < 1$. The boundary of the domain, determined using the method of generating functions, is shown in Fig. 4 by the dashed line. It is also calculated analytically. This is shown by the piecewise-smooth line which lies considerably below the boundary of existence of the parametric mapping $\min J = 0$.

Up to a chaotic state ($\epsilon < \epsilon_0(\alpha)$, $\epsilon_0(\alpha)$ is the critical value of the parameter ϵ), the phase portraits of the PP are easily investigated using the averaged Hamiltonian

$$\bar{H} = 3\epsilon^2 F(Y, \alpha)(1 - \cos X) + O(\epsilon^3)$$

calculated using formulae (6.6) and (9.3).

The PP lie on the invariant curves $F(Y, \alpha)(1 - \cos X) = \text{const}$. The phase portraits of the PP have the five topologically different structures shown in Fig. 5.

Structure 1, when $\alpha \in (-\infty, -1)$, is an anticlockwise motion along closed trajectories with a single fixed point.

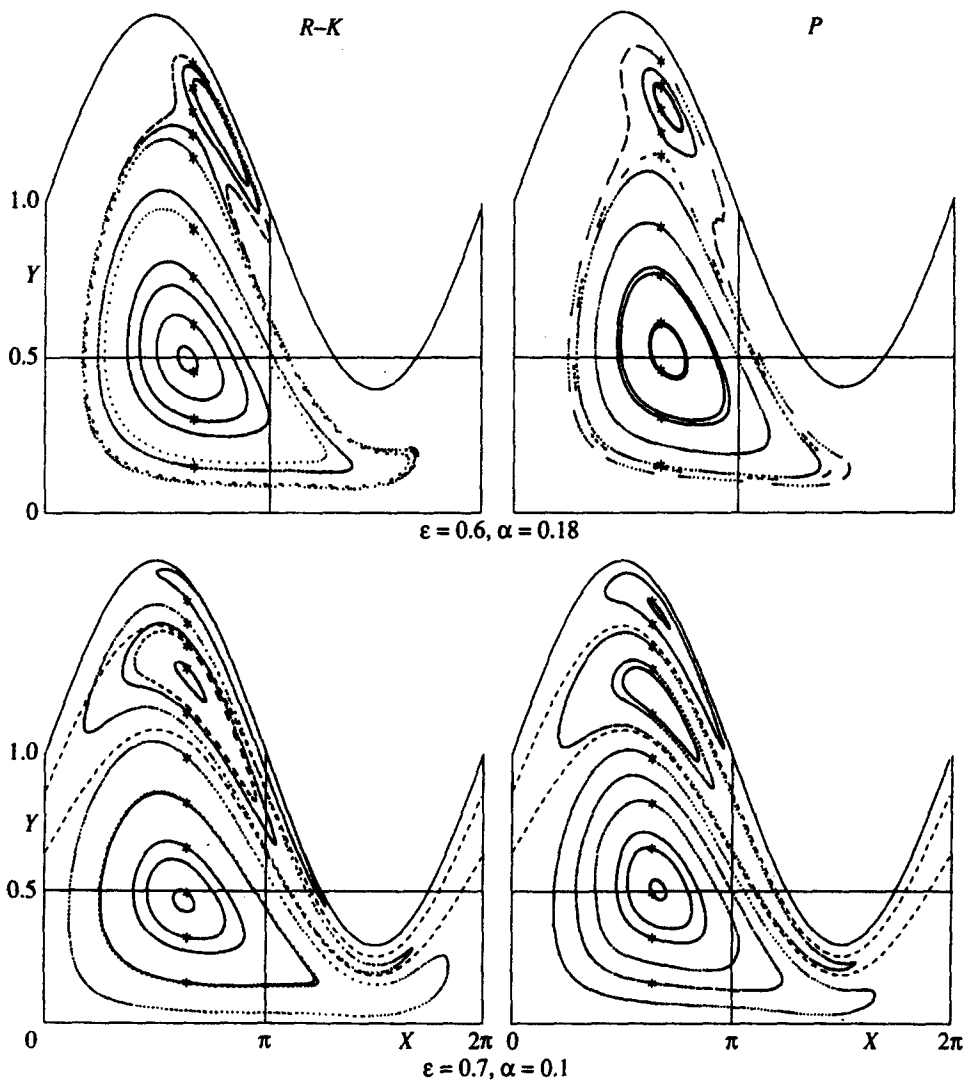


Fig. 6

Structure 2, when $\alpha \in (-1, 0)$, is two domains of motion separated by the separatrix $y = \frac{3}{4} - \sqrt{1 - 8\alpha}/4$, which is shown in Fig. 5 by the dashed line. In the upper domain there is anticlockwise motion along trajectories with a single fixed point. In the lower domain, there is clockwise motion along trajectories with a single fixed point.

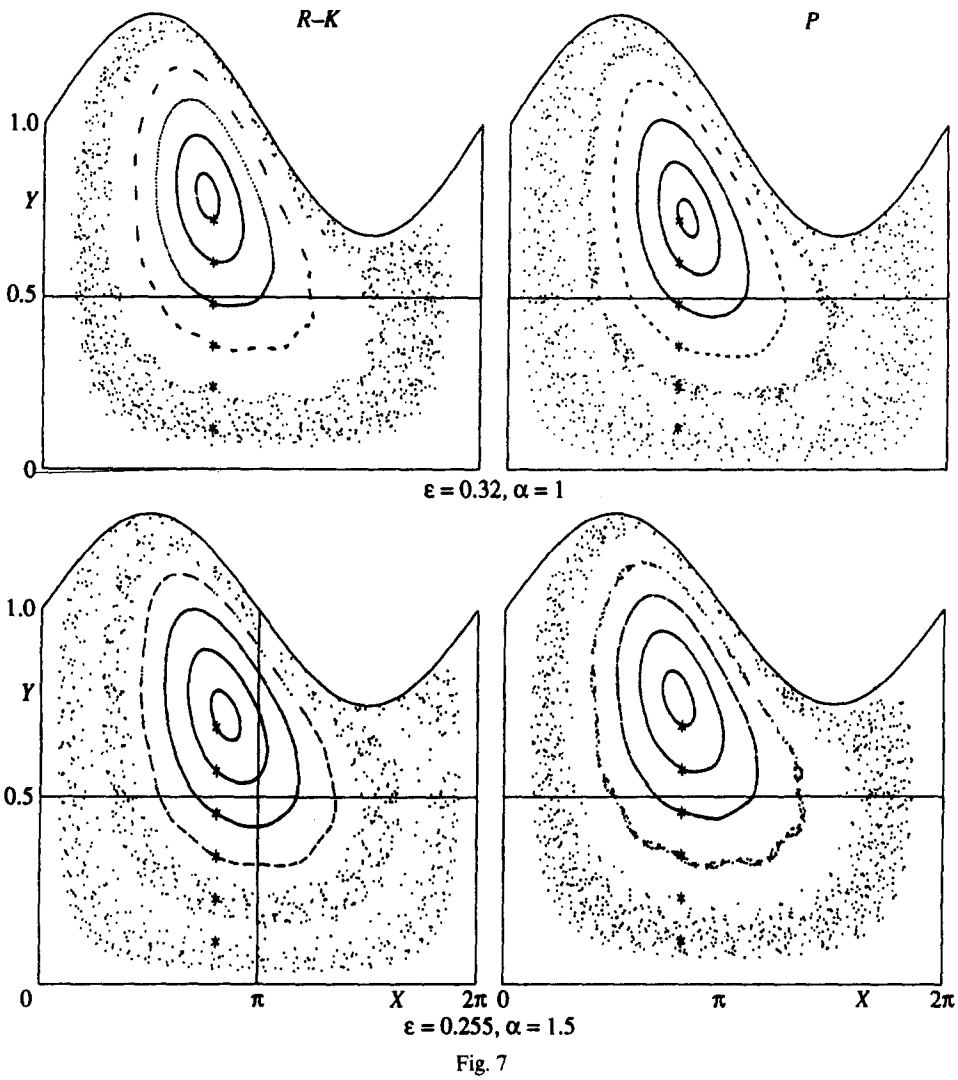
Structure 3, when $\alpha \in (0, \frac{1}{8})$, consists of three domains of motions separated by the two separatrices $y = \frac{3}{4} \pm \sqrt{1 - 8\alpha}/4$, which are shown by the dashed lines in Fig. 5. In the upper and lower domains, there is clockwise motion along trajectories with a single fixed point in each domain. In the middle domain, there is anticlockwise motion along trajectories with a single fixed point.

Structure 4, when $\alpha \in (\frac{1}{8}, 0.205)$, is a motion in three domains separated by separatrices. This is a unique case in which there is a fixed point of hyperbolic type corresponding to an unstable periodic trajectory. All the remaining fixed points in this and other cases correspond to stable periodic trajectories.

Structure 5, when $\alpha \in (0.205, \infty)$, is a clockwise motion along closed trajectories with a single fixed point.

When $\varepsilon < \varepsilon_0(\alpha)$, the PP found by direct solution of Hamilton's equations with Hamiltonian (9.1) move along trajectories, the topology of which corresponds to one of the five structures enumerated above. The type of structure is determined by the range in which the parameter α lies.

When the critical value of the parameter ε is exceeded, PP with the one and the same initial location can occupy a certain domain of the area S_i . We shall call the set of these domains the domain of chaotic motion. It has an area $S = \sum S_i$. We will now stipulate that the ratio $\sigma = S/S_0$, where S_0 is the area of the whole domain of motion, is considered as a measure of chaos. Domains of chaotic motion with a



different measure of chaos are shown in Fig. 4 in the domain of the parameters ϵ, α . The domain with a measure of chaos in the range $(1/4, 1/2)$ is marked by the oblique hatching, the domain with a measure of chaos in the range $(1/2, 3/4)$ is marked by the horizontal lines, the domain with a measure of chaos in the range greater than $3/4$ is shown by the points and, finally, the smooth background corresponds to a domain with a measure of chaos of less than $1/4$.

Calculation of the PP by the method of mappings (P) with the function (9.3) in the domain of existence of the mapping $J > 0$ is indistinguishable from the calculation of Hamilton's equations using the Runge-Kutta (R-K) method.

Examples of calculations are shown in Figs 6-8 for different values of ϵ and α . The initial positions of the points are denoted by asterisks. Their positions in a flow of 500 periods are shown by points.

Calculations of the position of the PP up to the transition to chaos are shown in Fig. 6. The topological structure of the PP in Fig. 6 corresponds to topological structure 4 ($\alpha \in (1/8, 0.205)$) (the upper part of Fig. 6) and to structure 3 ($\alpha \in (0, 1/8)$) (the lower part). In the lower part of Fig. 6, the separatrices $\bar{Y} = 3/4 \pm \sqrt{1 - 8\alpha/4}$ are shown by the dashed lines.

An example of calculations near the transition to chaos is shown in Fig. 7. All the notation is the same as in Fig. 6. The topological structure of the Poincaré points corresponds to the case when $\alpha > 0.205$. Both figures demonstrate the good agreement between the calculation using the parametric method and direct numerical calculation of the initial differential equations while, in the case of these values of the parameters α and ϵ , the asymptotic formulae according to the method of generating functions fall outside the limits of their applicability domain.

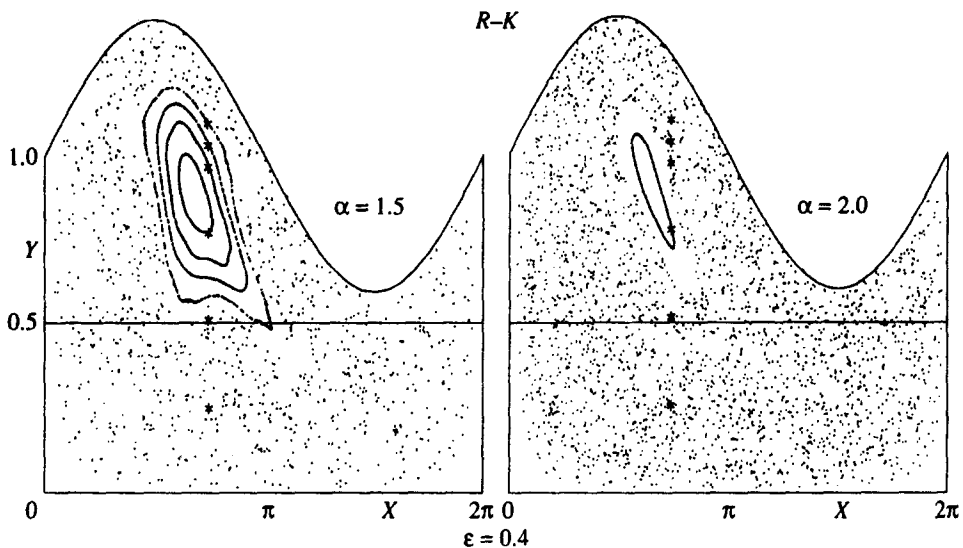


Fig. 8

Numerical calculations of PP, when the measure of chaos exceeds $3/4$, are shown in Fig. 8. This domain of parameters lies outside of the applicability domain of mapping with the function (9.3). For these values of the parameters, it is necessary to use a parametric mapping in the next approximation. In this domain of parameters, it is impossible to obtain a mapping by the method of generating functions in any approximation since, in the neighbourhood of the fixed point, the PP perform a complete rotation in less than after four periods in this case.

The example considered demonstrates the possibility of giving an analytical description of the transition to chaos in quite complex hydrodynamic systems. The parametric method of constructing the Poincaré mapping turns out to be particularly useful in problems with many parameters. The mappings calculated by this method have an acceptable accuracy even in the range of parameters when it is fundamentally impossible to obtain a Poincaré mapping using the classical method.

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